Double Wronskian Solution and Conservation Laws for a Generalized Variable-Coefficient Higher-Order Nonlinear Schrödinger Equation in Optical Fibers

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With applications in the higher-power and femtosecond optical transmission regime, a generalized variable-coefficient higher-order nonlinear Schrödinger (VC-HNLS) equation is analytically investigated. The multi-solitonic solutions of the generalized VC-HNLS equation in double Wronskian form is constructed and further verified using the Wronskian technique. Additionally, an infinite number of conservation laws for such an equation are presented. Finally, discussions and conclusions on results are made with figures plotted.

Key words: Generalized Variable-Coefficient Higher-Order Nonlinear Schrödinger Equation; Double Wronskian Solution; Wronskian Technique; Conservation Laws.

1. Introduction

Soliton theory [1, 2] in the nonlinear science plays an important role in various fields of science and engineering such as Bose-Einstein condensates, fluid mechanics, plasma physics, and nonlinear optics [3 – 16]. Many nonlinear phenomena can be described by the nonlinear evolution equations (NLEEs). To better understand those phenomena, many methods have been developed to find various analytic solutions, specially the soliton ones of NLEEs, such as the inverse scattering transformation [17, 18], bilinear method [19 – 21], Wronskian technique [22 – 24], and Darboux transformation [25]. Among these solitons, the optical solitons have currently attracted much interest for their potential applications in the long-haul optical communication systems or all-optical ultrafast switching devices and their unique properties of propagation without distortion and spreading [26, 27]. The dynamics of nonlinear optical pulse propagation in the picosecond regime are described by the nonlinear Schrödinger (NLS) equation with only the group velocity dispersion (GVD) and self-phase modulation (SPM). Many authors have focused their research on various soliton solutions of the NLS equation with uniform or nonuniform parameters theoretically and experimentally [28 – 31] (and references therein). However, when the optical pulse gets shorter, the NLS-type equations become inadequate. The governing equation of the ultra-short pulse propagation in the femtosecond domain, i. e. the higher-order NLS (HNLS) equation [32] was derived considering the effects of the transverse inhomogeneity and nonlinear dispersion and dissipation consistently to higher orders such as third-order dispersion (TOD), self-steepening (SS), and stimulate Raman scattering (SRS).

Nowadays, the investigation on the HNLS equation has been a topic of primary importance due to its significant applications in telecommunication and ultrafast signal-routing systems [33 – 35]. Considering real applications in the long-distance communications and manufacturing problems, there are more and more attention paid to the variable-coefficient HNLS (VC-HNLS) equations which can describe the pulse propagation in inhomogeneous fibers more realistically than the constant-coefficient ones [26, 27, 36 – 40]. Moreover, it is significant to study the dispersion management problem [41] described by the VC-

HNLS equation in the femtosecond regime. In this paper, a generalized VC-HNLS equation [42–44] is investigated,

$$\begin{aligned} &u_z+\alpha_1(z)u_t+\alpha_2(z)u+\mathrm{i}\alpha_3(z)u_{tt}+\alpha_4(z)u_{ttt}\\ &+\mathrm{i}\alpha_5(z)|u|^2u+\alpha_6(z)(|u|^2u)_t+\alpha_7(z)(|u|^2)_tu=0, \end{aligned} \tag{1}$$

which describes the femtosecond pulse propagation applicable to telecommunication and ultrafast signal-routing systems extensively in the weakly dispersive and nonlinear dielectrics with distributed parameters. The function $u=u(z,t)$ is the complex envelope of the electrical field in the monomode optical fiber with respect to the propagation distance z and the time t. The term proportional to $\alpha_1(z)$ results from the group velocity. $\alpha_2(z)$ is related to the heat-insulating amplification or loss. $\alpha_3(z)$ and $\alpha_4(z)$ represent the effects of GVD and TOD, respectively. $\alpha_5(z)$ is the SPM parameter, and the parameters $\alpha_6(z)$ and $\alpha_7(z)$ denote the effects of SS and SRS, respectively. All the coefficients $\alpha_j(z)$ $(j=1,2,\cdots,7)$ are real functions of z.

One representation associated with multi-soliton solutions is Wronskian which was first introduced by Satsuma [45]. Furthermore, Freeman and Nimmo developed the Wronskian technique [46–48], a remarkable feature of which is that the Wronskian solution can be verified by direct substitution into the bilinear form of the NLEE [49], since the differentiation of this kind of determinant leads to the sum of a number of determinants relying not on the size of the determinant, but merely upon the number of derivatives. In the present paper, the multi-solitonic solutions of (1) in double Wronskian form are presented on the basis of the Lax pair under special coefficient constraints and verified by virtue of the Wronskian technique. Furthermore, as one of the integrable properties for the soliton equations, an infinite number of conservation laws are presented which assures the completely integrability of (1) under special coefficient constraints.

The organization of this paper is as follows: In Section 2, the double Wronskian solution of (1) is constructed under special coefficient constraints. And then, making use of the Wronskian technique, the verification of the double Wronskian solution is given by direct substitution into the bilinear form. In Section 3, an infinite number of conservation laws are presented. Section 4 is devoted to discussions and conclusions on the results and the graphical illustrations for solitonic solutions of (1).

2. Double Wronskian Solution

Referred to [44], (1) is completely integrable in the sense of possessing Lax pair under the coefficient constraints,

$$\alpha_2(z)=\frac{\alpha_5'(z)\alpha_3(z)-\alpha_3'(z)\alpha_5(z)}{2\alpha_5(z)\alpha_3(z)}, \tag{2}$$

$$3\alpha_4(z)\alpha_5(z)=\alpha_3(z)\left[3\alpha_6(z)+2\alpha_7(z)\right], \tag{3}$$

$$\alpha_6(z)+\alpha_7(z)=0. \tag{4}$$

Actually, constraint (2) can be reduced to

$$\frac{\alpha_5(z)}{\alpha_3(z)}=c_0\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}, \tag{5}$$

with c_0 as an arbitrary nonzero real integration constant. The following Lax pair of (1) have been derived by the authors:

$$\begin{aligned} &\phi_t=\mathbf{U}\phi,\\ &\mathbf{U}=\begin{pmatrix}\lambda & \beta(z)u(z,t)\\ \gamma(z)u^*(z,t) & -\lambda\end{pmatrix}, \end{aligned} \tag{6}$$

$$\begin{aligned} &\phi_z=\mathbf{V}\phi,\\ &\mathbf{V}=\begin{pmatrix}A(z,t,\lambda) & B(z,t,\lambda)\\ C(z,t,\lambda) & -A(z,t,\lambda)\end{pmatrix},\\ &\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \end{aligned} \tag{7}$$

where $u(z,t)$ is the potential, λ is the spectral parameter and

$$\begin{aligned} A=&-4\alpha_4(z)\lambda^3-2\mathrm{i}\alpha_3(z)\lambda^2\\ &-\left[\frac{\alpha_4(z)\alpha_5(z)}{\alpha_3(z)}|u|^2+\alpha_1(z)\right]\lambda-\mathrm{i}\frac{\alpha_5(z)}{2}|u|^2\\ &+\frac{\alpha_4(z)\alpha_5(z)}{2\alpha_3(z)}(uu_t^*-u^*u_t)+a_0(z), \end{aligned}$$

$$\begin{aligned} B=\beta(z)\Big\{&-4\alpha_4(z)u\lambda^2-[2\mathrm{i}\alpha_3(z)u\\ &+2\alpha_4(z)u_t]\lambda-\frac{\alpha_4(z)\alpha_5(z)}{\alpha_3(z)}|u|^2u\\ &-\mathrm{i}\alpha_3(z)u_t-\alpha_4(z)u_{tt}-\alpha_1(z)u\Big\}, \end{aligned}$$

$$\begin{aligned} C=\beta(z)\Big\{&4\alpha_4(z)u^*\lambda^2+[2\mathrm{i}\alpha_3(z)u^*\\ &-2\alpha_4(z)u_t^*]\lambda+\frac{\alpha_4(z)\alpha_5(z)}{\alpha_3(z)}|u|^2u^*\\ &-\mathrm{i}\alpha_3(z)u_t^*+\alpha_4(z)u_{tt}^*+\alpha_1(z)u^*\Big\}, \end{aligned}$$

where $a_0(z)$ is a function of integration and $\beta(z) = -\gamma(z) = \sqrt{\frac{\alpha_5(z)}{2\alpha_3(z)}}$. The compatibility condition $\mathbf{U}_z - \mathbf{V}_t + [\mathbf{U},\mathbf{V}]=0$ gives rise to (1) under constraints (2–4), which means that (1) is completely integrable.

In the following, the double Wronskian solution will be presented and verified via the Wronskian technique. Through the dependent variable transformation

$$u(z,t) = \kappa(z)\frac{g(z,t)}{f(z,t)}, \quad (8)$$

where $g(z,t)$ is a complex function and $f(z,t)$ is a real one, the resulting bilinear form of (1) [50] is obtained under constraints (3) and (4),

$$\left[D_z + \alpha_1(z)D_t + \mathrm{i}\alpha_3(z)D_t^2 + \alpha_4(z)D_t^3\right](g\cdot f) = 0, \quad (9)$$

$$\alpha_3(z)D_t^2(f\cdot f) = \alpha_5(z)\kappa(z)^2|g|^2, \quad (10)$$

where $\kappa(z)=c_1\mathrm{e}^{-\int\alpha_2(z)\mathrm{d}z}$ with c_1 as an arbitrary constant and D_x and D_t are the bilinear derivative operators [49] defined as

$$D_x^m D_t^n(a\cdot b) = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')\bigg|_{x'=x,t'=t}.$$

Under constraint (2) with $c_0c_1^2 = 2$, (10) becomes

$$D_t^2(f\cdot f) = 2|g|^2. \quad (11)$$

The double Wronskian determinant is defined as

$$W^{N,M}(\varphi,\psi) = \det\left(\varphi,\partial_t\varphi,\cdots,\partial_t^{N-1}\varphi;\ \psi,\partial_t\psi,\cdots,\partial_t^{M-1}\psi\right),$$

with $\varphi = (\varphi_1,\varphi_2,\cdots,\varphi_{N+M})^T$ and $\psi = (\psi_1,\psi_2,\cdots,\psi_{N+M})^T$ where T denotes the vector transpose. For convenience, the double Wronskian determinant is denoted in the abbreviated notation

$$W^{N,M}(\varphi,\psi) = (\widehat{N-1};\widehat{M-1}).$$

By virtue of the Lax pair of (1), we suppose

$$\begin{aligned} g &= 2W^{N+1,N-1}(\varphi,\psi) = 2(\widehat{N};\widehat{N-2}),\\ g^* &= 2W^{N-1,N+1}(\varphi,\psi) = 2(\widehat{N-2};\widehat{N}),\\ f &= W^{N,N}(\varphi,\psi) = (\widehat{N-1};\widehat{N-1}), \end{aligned} \quad (12)$$

where

$$\varphi_j = \mathrm{e}^{\varepsilon_j},\quad \psi_j = \mathrm{e}^{-\varepsilon_j},\quad \varphi_{j+N} = -\psi_j^* = -\mathrm{e}^{-\varepsilon_j^*},$$
$$\psi_{j+N} = \varphi_j^* = \mathrm{e}^{\varepsilon_j^*},\quad (j=1,2,\cdots,N)$$

with

$$\varepsilon_j = k_jt - k_j\int\alpha_1(z)\mathrm{d}z - 2\mathrm{i}k_j{}^2\int\alpha_3(z)\mathrm{d}z - 4k_j{}^3\int\alpha_4(z)\mathrm{d}z + \varepsilon_{j0},$$

where k_j and $\varepsilon_{j0}(j=1,2,\cdots,N)$ are complex constants.

Employing the Wronskian technique, it can be proved that f and g defined in the double Wronskian form indeed satisfy (9) and (11). Firstly, the derivatives of f with respect to t and z are given as below,

$$f_t = (\widehat{N-2},N;\widehat{N-1}) + (\widehat{N-1};\widehat{N-2},N),$$

$$\begin{aligned} f_{tt} = {}&(\widehat{N-3},N-1,N;\widehat{N-1}) + (\widehat{N-2},N+1;\widehat{N-1})\\ &+2(\widehat{N-2},N;\widehat{N-2},N) + (\widehat{N-1};\widehat{N-3},N-1,N)\\ &+(\widehat{N-1};\widehat{N-2},N+1), \end{aligned}$$

$$\begin{aligned} f_{ttt} = {}&(\widehat{N-4},N-2,N-1,N;\widehat{N-1}) + 2(\widehat{N-3},N-1,N+1;\widehat{N-1}) + 3(\widehat{N-3},N-1,N;\widehat{N-2},N)\\ &+3(\widehat{N-2},N+1;\widehat{N-2},N) + (\widehat{N-2},N+2;\widehat{N-1})\\ &+(\widehat{N-1};\widehat{N-4},N-2,N-1,N) + 2(\widehat{N-1};\widehat{N-3},N-1,N+1) + 3(\widehat{N-2},N;\widehat{N-3},N-1,N)\\ &+3(\widehat{N-2},N;\widehat{N-2},N+1) + (\widehat{N-1};\widehat{N-2},N+2), \end{aligned}$$

$$\begin{aligned} f_z = {}&-\alpha_1(z)\left[(\widehat{N-2},N;\widehat{N-1}) + (\widehat{N-1};\widehat{N-2},N)\right]\\ &+2\mathrm{i}\alpha_3(z)\Big[(\widehat{N-3},N-1,N;\widehat{N-1}) - (\widehat{N-2},N+1;\widehat{N-1}) - (\widehat{N-1};\widehat{N-3},N-1,N) + (\widehat{N-1};\widehat{N-2},N+1)\Big] - 4\alpha_4(z)\Big[(\widehat{N-4},N-2,N-1,N;\widehat{N-1})\\ &-(\widehat{N-3},N-1,N+1;\widehat{N-1}) + (\widehat{N-2},N+2;\widehat{N-1})\\ &+(\widehat{N-1};\widehat{N-4},N-2,N-1,N) - (\widehat{N-1};\widehat{N-3},N-1,N+1) + (\widehat{N-1};\widehat{N-2},N+2)\Big]. \end{aligned}$$

The corresponding derivatives and identities related to g can be obtained similarly.

Substituting various derivatives of f and g into (9) and (11), and utilizing the determinant identities in the Appendix yield,

$$
\begin{aligned}
&\left[D_z+\alpha_1(z)D_t+\mathrm{i}\alpha_3(z)D_t^2+\alpha_4(z)D_t^3\right](g\cdot f)= \\
&4\mathrm{i}\alpha_3(z)\left[\begin{vmatrix} \widehat{N-2} & 0 & \widehat{\mathbf{N-2}} & \mathbf{0} & N-1 & N & N+1 & \mathbf{N-1} \\ 0 & \widehat{N-2} & \mathbf{0} & \widehat{\mathbf{N-2}} & N-1 & N & N+1 & \mathbf{N-1} \end{vmatrix}\right. \\
&\qquad\left.+\begin{vmatrix} \widehat{N-1} & 0 & \widehat{\mathbf{N-3}} & \mathbf{0} & N & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N} \\ 0 & \widehat{N-1} & \mathbf{0} & \widehat{\mathbf{N-3}} & N & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N} \end{vmatrix}\right] \\
&+6\alpha_4(z)\left[\begin{vmatrix} \widehat{N-1} & 0 & \widehat{\mathbf{N-4}} & \mathbf{N-2} & \mathbf{0} & \mathbf{0} & N & \mathbf{N-3} & \mathbf{N-1} & \mathbf{N} \\ 0 & \widehat{N-1} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{N-4}} & \mathbf{N-2} & N & \mathbf{N-3} & \mathbf{N-1} & \mathbf{N} \end{vmatrix}\right. \\
&\qquad+\begin{vmatrix} \widehat{N-2} & 0 & \widehat{\mathbf{N-2}} & \mathbf{0} & N-1 & N & N+2 & \mathbf{N-1} \\ 0 & \widehat{N-2} & \mathbf{0} & \widehat{\mathbf{N-2}} & N-1 & N & N+2 & \mathbf{N-1} \end{vmatrix} \\
&\qquad-\begin{vmatrix} \widehat{N-3} & N-1 & 0 & 0 & \widehat{\mathbf{N-2}} & \mathbf{0} & N-2 & N & N+1 & \mathbf{N-1} \\ 0 & 0 & \widehat{N-3} & N-1 & \mathbf{0} & \widehat{\mathbf{N-2}} & N-2 & N & N+1 & \mathbf{N-1} \end{vmatrix} \\
&\qquad+\begin{vmatrix} \widehat{N-2} & 0 & \widehat{\mathbf{N-3}} & \mathbf{N-1} & \mathbf{0} & \mathbf{0} & N-1 & N & N+1 & \mathbf{N-2} \\ 0 & \widehat{N-2} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{N-3}} & \mathbf{N-1} & N-1 & N & N+1 & \mathbf{N-2} \end{vmatrix} \\
&\qquad+\begin{vmatrix} \widehat{N-1} & 0 & \widehat{\mathbf{N-3}} & \mathbf{0} & N+1 & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N} \\ 0 & \widehat{N-1} & \mathbf{0} & \widehat{\mathbf{N-3}} & N+1 & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N} \end{vmatrix} \\
&\qquad-\begin{vmatrix} \widehat{N-1} & 0 & \widehat{\mathbf{N-3}} & \mathbf{0} & N & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N+1} \\ 0 & \widehat{N-1} & \mathbf{0} & \widehat{\mathbf{N-3}} & N & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N+1} \end{vmatrix} \\
&\qquad-\begin{vmatrix} \widehat{N-2} & N & 0 & 0 & \widehat{\mathbf{N-3}} & \mathbf{0} & N-1 & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N} \\ 0 & 0 & \widehat{N-2} & N & \mathbf{0} & \widehat{\mathbf{N-3}} & N-1 & \mathbf{N-2} & \mathbf{N-1} & \mathbf{N} \end{vmatrix} \\
&\qquad\left.-\begin{vmatrix} \widehat{N-2} & 0 & \widehat{\mathbf{N-2}} & \mathbf{0} & N-1 & N & N+1 & \mathbf{N} \\ 0 & \widehat{N-2} & \mathbf{0} & \widehat{\mathbf{N-2}} & N-1 & N & N+1 & \mathbf{N} \end{vmatrix}\right]=0,
\end{aligned}
$$

and

$$
D_t^2(f\cdot f)-2|g|^2=-4\begin{vmatrix} \widehat{N-2} & 0 & \widehat{\mathbf{N-2}} & \mathbf{0} & N-1 & N & \mathbf{N-1} & \mathbf{N} \\ 0 & \widehat{N-2} & \mathbf{0} & \widehat{\mathbf{N-2}} & N-1 & N & \mathbf{N-1} & \mathbf{N} \end{vmatrix}=0,
$$

where the bold type denotes the contributions from the second half of the determinant. Up to now, we have proved that f and g defined in the double Wronskian form indeed satisfy (9) and (11). As sample application with $c_1=1$, the bright one-solitonic solution is given as

$$
u=2(k_1+k_1^*)\mathrm{e}^{-\int\alpha_2(z)\mathrm{d}z}\frac{\mathrm{e}^{\varepsilon_1-\varepsilon_1^*}}{\mathrm{e}^{\varepsilon_1+\varepsilon_1^*}+\mathrm{e}^{-\varepsilon_1-\varepsilon_1^*}}=2k_{1\mathrm{R}}\mathrm{e}^{-\int\alpha_2(z)\mathrm{d}z}\operatorname{sech}(2\varepsilon_{1\mathrm{R}})\mathrm{e}^{\mathrm{i}\varepsilon_{1\mathrm{I}}}, \tag{13}
$$

where $k_{1\mathrm{R}}$ is the real part of k_1 with $\varepsilon_{1\mathrm{R}}$ and $\varepsilon_{1\mathrm{I}}$ as the real and imaginary parts of ε_1. And the bright two-solitonic solution can be derived as below

$$
u=\frac{\Lambda_1\cosh(2\varepsilon_{2\mathrm{R}})\mathrm{e}^{2\mathrm{i}\varepsilon_{1\mathrm{I}}}+\Lambda_2\cosh(2\varepsilon_{1\mathrm{R}})\mathrm{e}^{2\mathrm{i}\varepsilon_{2\mathrm{I}}}+\Lambda_3\left[\sinh(2\varepsilon_{2\mathrm{R}})\mathrm{e}^{2\mathrm{i}\varepsilon_{1\mathrm{I}}}-\sinh(2\varepsilon_{1\mathrm{R}})\mathrm{e}^{2\mathrm{i}\varepsilon_{2\mathrm{I}}}\right]}{\Upsilon_1\cosh(2\varepsilon_{1\mathrm{R}}+2\varepsilon_{2\mathrm{R}})+\Upsilon_2\cosh(2\varepsilon_{1\mathrm{R}}-2\varepsilon_{2\mathrm{R}})+\Upsilon_3\cos(\varepsilon_{1\mathrm{I}}-\varepsilon_{2\mathrm{I}})}\mathrm{e}^{-\int\alpha_2(z)\mathrm{d}z}, \tag{14}
$$

where

$$
\Lambda_1=2k_{1\mathrm{R}}\left[k_{1\mathrm{R}}^2-k_{2\mathrm{R}}^2+(k_{1\mathrm{I}}-k_{2\mathrm{I}})^2\right],\quad \Lambda_2=2k_{1\mathrm{R}}\left[-k_{1\mathrm{R}}^2+k_{2\mathrm{R}}^2+(k_{1\mathrm{I}}-k_{2\mathrm{I}})^2\right],\quad \Lambda_3=4\mathrm{i}k_{1\mathrm{R}}k_{2\mathrm{R}}\left(k_{1\mathrm{I}}-k_{2\mathrm{I}}\right),
$$

$$
\Upsilon_1=\frac{1}{2}\left[(k_{1\mathrm{R}}-k_{2\mathrm{R}})^2+(k_{1\mathrm{I}}-k_{2\mathrm{I}})^2\right],\quad \Upsilon_2=\frac{1}{2}\left[(k_{1\mathrm{R}}+k_{2\mathrm{R}})^2+(k_{1\mathrm{I}}-k_{2\mathrm{I}})^2\right],\quad \Upsilon_3=8k_{1\mathrm{R}}k_{2\mathrm{R}},
$$

with k_{jR} and k_{jI} as the real and imaginary parts of $k_j (j=1,2)$, and ε_{jR} and ε_{jI} are the real and imaginary parts of k_j and $\varepsilon_j (j=1,2)$.

3. Conservation Laws

In the following, with the aid of the Lax pair, an infinite number of conservation laws for (1) can be derived.

Supposing

$$\Gamma = \frac{\phi_2}{\phi_1}, \tag{15}$$

Equation (6) can be written as the following Γ-Riccati form:

$$\Gamma_t = -\beta(z)u^* - 2\lambda\Gamma - \beta(z)u\Gamma^2. \tag{16}$$

Noting

$$W = \beta(z)u\Gamma = \sum_{n=1}^{\infty} \frac{\omega_n(z,t)}{(2\lambda)^n}, \tag{17}$$

and substituting the expansion into (16) yield

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\omega_n}{(2\lambda)^n} + \frac{\beta(z)^2 uu^*}{2\lambda} + \sum_{n=2}^{\infty} \frac{\omega_{n-1,t}}{(2\lambda)^n} \\ &-\frac{u_t}{u}\sum_{n=2}^{\infty} \frac{\omega_{n-1}}{(2\lambda)^n} + \sum_{n=2}^{\infty} \frac{\sum_{k=1}^{n-1}\omega_k\omega_{n-1-k}}{(2\lambda)^n} = 0. \end{aligned} \tag{18}$$

According to the coefficients equations of the equal powers of 2λ, the recurrence relation can be obtained as below:

$$\omega_1 = -\beta(z)^2 uu^*, \tag{19}$$

$$\omega_n = -\omega_{n-1,t} + \frac{u_t}{u}\omega_{n-1} - \sum_{k=1}^{n-1} \omega_k\omega_{n-1-k}. \quad (n=2,3,\cdots) \tag{20}$$

Considering the consistent relation $(\ln\phi_1)_{tz} = (\ln\phi_1)_{zt}$, we can get the following conservative form for (1):

$$(\lambda + W)_z = \left[A + \frac{B}{\beta(z)u}W\right]_t, \tag{21}$$

where $\lambda_z = 0$ corresponding to the isospectral condition with A and B are defined above in the Lax pair. Inserting W, A and B into (21) and equating the equal power of 2λ to be zero, we can get an infinite number of conservation laws in the form

$$\frac{\partial T_k}{\partial z} + \frac{\partial X_k}{\partial t} = 0, \tag{22}$$

where T_k and X_k are called conserved density and flux, respectively. Here, we list the first three conservation laws:

$$T_1 = -c_0 \mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z} uu^*, \tag{23}$$

$$\begin{aligned} X_1 = c_0 \mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}\Big[&\alpha_1(z)|u|^2 + \frac{3}{2}c_0\alpha_4(z)\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}|u|^4 \\ &+ \mathrm{i}\alpha_3(z)(u^*u_t - uu_t^*) \\ &+ \alpha_4(z)(uu_{tt}^* - u_tu_t^* + u_{tt}u^*)\Big], \end{aligned} \tag{24}$$

$$T_2 = c_0 \mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z} uu_t^*, \tag{25}$$

$$\begin{aligned} X_2 = c_0 \mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}\Big[&\frac{\mathrm{i}}{2}\alpha_5(z)|u|^4 - \alpha_1(z)uu_t^* \\ &- 3c_0\alpha_4(z)\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}u^2u^*u_t^* \\ &+ \mathrm{i}\alpha_3(z)(uu_{tt}^* - u_tu_t^*) \\ &- \alpha_4(z)(uu_{ttt}^* - u_tu_{tt}^* + u_{tt}u_t^*)\Big], \end{aligned} \tag{26}$$

$$T_3 = -c_0 \mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}\left[\frac{1}{2}c_0\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}|u|^4 + uu_{tt}^*\right], \tag{27}$$

$$\begin{aligned} X_3 = c_0 \mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}\Big\{&\frac{1}{2}c_0\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}\big[\alpha_1(z)|u|^2 \\ &- \alpha_4(z)u_t^2u^{*2} + 5\alpha_4(z)u^2u_t^{*2}\big] \\ &+ c_0\alpha_4(z)\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}|u|^2\big[c_0\mathrm{e}^{2\int\alpha_2(z)\mathrm{d}z}|u|^4 \\ &+ u_{tt}u^* + u_tu_t^* + 4uu_{tt}^*\big] + \alpha_1(z)uu_{tt}^* \\ &- 2\mathrm{i}\alpha_5(z)u^2u^*u_t^* + \mathrm{i}\alpha_3(z)\big[u_tu_{tt}^* - uu_{ttt}^*\big] \\ &+ \alpha_4(z)\big[u_{tt}u_{tt}^* - u_tu_{ttt}^* + uu_{tttt}^*\big]\Big\}. \end{aligned} \tag{28}$$

4. Discussions and Conclusions

Optical solitons in fibers have attracted much interest for their potential applications in the long-haul optical communication systems or all-optical ultrafast switching devices and their unique properties of propagation without distortion and spreading. They may become the ideal information carriers in long-distance communications. The femtosecond pulse propagation is governed by the HNLS equation with the effects of TOD, SS and SRS. Considering real applications in the long-distance communications and manufacturing

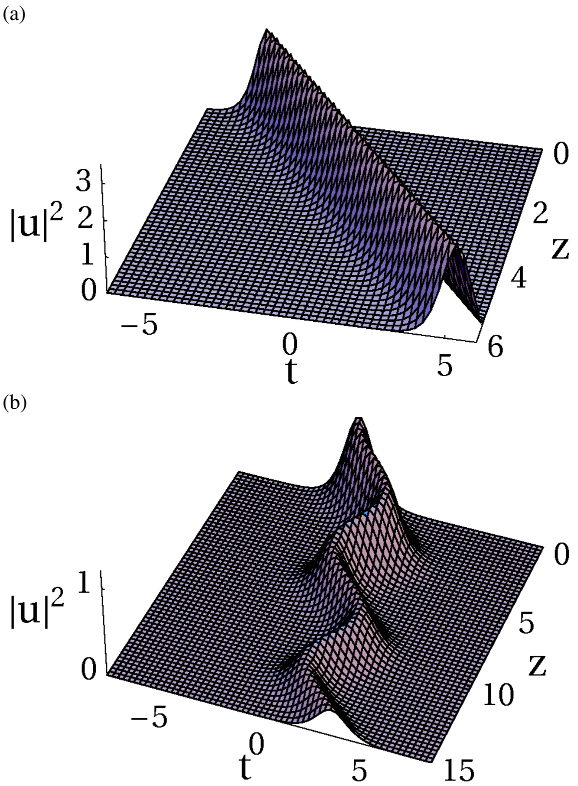

Fig. 1. The intensity evolution plot of the bright one-solitonic solution with parameters: (a): $k_1 = 0.8$, $\varepsilon_{10} = 3$, $\alpha_1(z) = 1$, $\alpha_2(z) = 0$, $\alpha_3(z) = 0.5$, and $\alpha_4(z) = 0.2$; (b): $k_1 = 0.5$, $\varepsilon_{10} = 0.3+0.2\text{i}$, $\alpha_1(z) = \sin z$, $\alpha_2(z) = 0.04$, $\alpha_3(z) = 1+\sin z$, and $\alpha_4(z) = 0.2$.

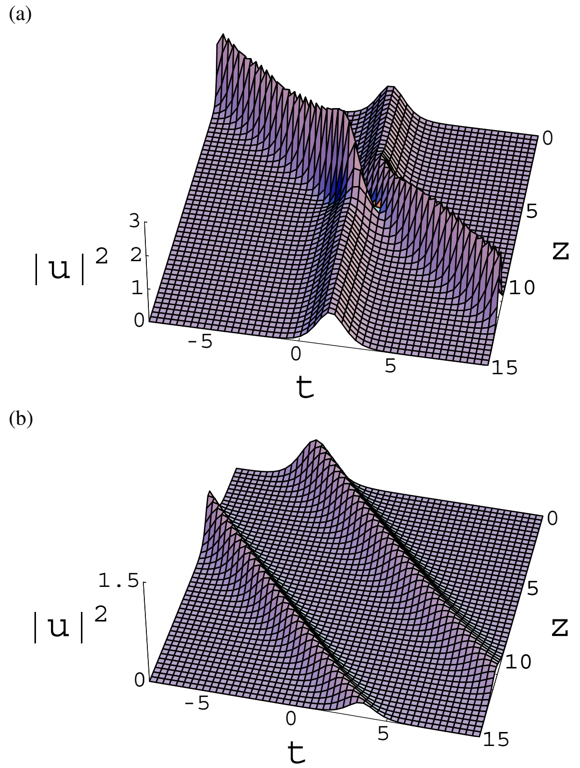

Fig. 2. The head-on evolution plot of the bright two-solitonic solution with parameters: (a): $k_1 = 0.5+0.6\text{i}$, $k_2 = 0.7$, $\varepsilon_{10} = -0.5-2\text{i}$, $\varepsilon_{20} = 5$, $\alpha_1(z) = 1$, $\alpha_2(z) = 0$, $\alpha_3(z) = 0.1$, and $\alpha_4(z)$=0.2; (b): the over-taking evolution plot of the bright two-solitonic solution with the same parameters with (a) except $k_1 = 0.5+0.1\text{i}$, $k_2 = 0.4-0.1\text{i}$, $\varepsilon_{10} = -6-2\text{i}$, $\varepsilon_{20} = 2$, and $\alpha_2(z) = 0.05$.

problems, a generalized VC-HNLS equation, i. e. (1), has been analytically investigated under special coefficients constraints in this paper.

With the help of the Lax pair for (1) under constraints (2) – (4), the bright N-solitonic solution in double Wronskian form of (1) has been constructed and verified by direct substitution into the bilinear form, i. e. (9) and (11), via the Wronskian technique. Associated with the complete integrability of (1) in the sense of possessing Lax pair under special constraints, an infinite number of conservation laws can be derived with the first three ones listed. Actually, there are many authors who have studied constraints (2) – (4) called the generalized Hirota condition from mathematical and physical viewpoints [36 – 40, 42, 51] (and references therein). Constraints (2) – (4) provide conditions for (1) to be completely integrable. As a completely integrable model under the special constraints, (1) has many good properties such as multi-solitonic solutions and an infinite number of conservation laws which have been obtained in this paper.

For the sample bright one-solitonic solution (13), the evolution of its intensity is

$$|u|^2 = 4{k_{1\text{R}}}^2 \text{e}^{-2\int \alpha_2(z)\text{d}z} \operatorname{sech}^2(2\varepsilon_{1\text{R}}). \tag{29}$$

Supposing $\text{Re}\, k_1 > 0$, with vanishing boundary condition for the bright one-solitonic solution (13), it can be found that

$$\int_{-\infty}^{+\infty} \text{e}^{2\int \alpha_2(z)\text{d}z} |u|^2 \text{d}t = 4k_{1\text{R}}, \tag{30}$$

which indicates that the energy $\int_{-\infty}^{+\infty} |u|^2 \text{d}t$ will exponentially decay/grow as the rate $\text{e}^{2\int \alpha_2(z)\text{d}z}$. From Expression (29), $\alpha_2(z)$ is a primary factor affecting the intensities of the solitary waves and the wave velocity $\alpha_1(z) - 4k_{1\text{I}}\alpha_3(z) + 4\alpha_4(z){k_{1\text{R}}}^2 - 12{k_{1\text{I}}}^2\alpha_4(z)$ can be influenced by $\alpha_1(z)$, $\alpha_3(z)$ and $\alpha_4(z)$. Figures 1(a) and 1(b) depict the intensity evolutions of the bright one-solitonic solutions with different parameters. The intensity of the bright soliton in Figure 1(a) keeps invariant without perturbation while the one in Figure 1(b) undergoes the attenuation and periodic oscillation with $\alpha_2(z)$ as a nonzero constant and trigonometric function $\alpha_1(z)$ and the periodic dispersion $\alpha_3(z)$. Figure 2(a) shows an elastic head-on collision of two bright solitons with a phase shift at the moment of interaction while the two over-taking solitons in Figure 2(b) even do not interact for a long propagation distance with attenuating amplitudes.

In conclusion, the generalized VC-HNLS equation which describes the pulse propagation in the femtosecond regime has been investigated analytically in this paper. By virtue of the Wronskian technique, the bright solitonic solutions in double Wronskian form of the generalized VC-HNLS equation have been constructed with the help of the Lax pair under certain coefficient constraints, and verified by direct substitution into its bilinear form. Additionally, an infinite number of conservation laws have been derived for the generalized VC-HNLS equation. It can be expected that the techniques used in this paper can also be used to investigate the integrable properties of several other NLEEs with variable coefficients. The constraints under which the solitonic solutions are derived, may be helpful for studying the soliton propagation and dispersion management systems theoretically and experimentally.

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Appendix A

The following Wronskian determinant identities are utilized in the proof: process,

$$\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N-1};\widehat{N-1}) = (\widehat{N-2},N;\widehat{N-1}) - (\widehat{N-1};\widehat{N-2},N),$$

$$\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^2(\widehat{N-1};\widehat{N-1}) = (\widehat{N-3},N-1,N;\widehat{N-1}) + (\widehat{N-2},N+1;\widehat{N-1})$$
$$-2(\widehat{N-2},N;\widehat{N-2},N) + (\widehat{N-1};\widehat{N-3},N-1,N) + (\widehat{N-1};\widehat{N-2},N+1),$$

$$\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^3(\widehat{N-1};\widehat{N-1}) = (\widehat{N-4},N-2,N-1,N;\widehat{N-1}) + 2(\widehat{N-3},N-1,N+1;\widehat{N-1})$$
$$-3(\widehat{N-3},N-1,N;\widehat{N-2},N) - 3(\widehat{N-2},N+1;\widehat{N-2},N) + (\widehat{N-2},N+2;\widehat{N-1})$$
$$-(\widehat{N-1};\widehat{N-4},N-2,N-1,N) - 2(\widehat{N-1};\widehat{N-3},N-1,N+1) + 3(\widehat{N-2},N;\widehat{N-3},N-1,N)$$
$$+3(\widehat{N-2},N;\widehat{N-2},N+1) - (\widehat{N-1};\widehat{N-2},N+2),$$

$$(\widehat{N-1};\widehat{N-1})\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^2(\widehat{N-1};\widehat{N-1})\right\} = \left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N-1};\widehat{N-1})\right\}^2,$$

$$(\widehat{N};\widehat{N-2})\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^3(\widehat{N-1};\widehat{N-1})\right\}=(\widehat{N-1};\widehat{N-1})\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^3(\widehat{N};\widehat{N-2})\right\}$$
$$=\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^2(\widehat{N};\widehat{N-2})\right\}\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N-1};\widehat{N-1})\right\}$$
$$=\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right]^2(\widehat{N-1};\widehat{N-1})\right\}\left\{\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N};\widehat{N-2})\right\},$$

$$\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N-1},N+1;\widehat{N-3},N-1)=(\widehat{N-2},N,N+1;\widehat{N-3},N-1)$$
$$=(\widehat{N-1},N+2;\widehat{N-3},N-1)-(\widehat{N-1},N+1;\widehat{N-4},N-2,N-1)-(\widehat{N-1},N+1;\widehat{N-3},N),$$
$$\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N};\widehat{N-4},N-2,N-1)=(\widehat{N-1},N+1;\widehat{N-4},N-2,N-1)$$
$$-(\widehat{N};\widehat{N-5},N-3,N-2,N-1)-(\widehat{N};\widehat{N-4},N-2,N),$$
$$\left[\sum_{j=1}^{N}(k_j-k_j^*)\right](\widehat{N-2},N,N+1;\widehat{N-2})=(\widehat{N-3},N-1,N,N+1;\widehat{N-2})$$
$$+(\widehat{N-2},N,N+2;\widehat{N-2})-(\widehat{N-2},N,N+1;\widehat{N-3},N-1).$$

The following two determinant identities are also be used:

$$(1)\quad |D,a,b||D,c,d|-|D,a,c||D,b,d|+|D,a,d||D,b,c|=0,$$

where D is an $N\times(N-2)$ matrix with a,b,c and d representing N-dimensional column vectors.

$$(2)\quad \sum_{j=1}^{N}|a_1,\cdots,a_{j-1},ba_j,a_{j+1},\cdots,a_N|=\left(\sum_{j=1}^{N}b_j\right)|a_1,\cdots,a_N|,$$

where a_j are N-dimensional column vectors and ba_j represent $(b_1a_{1j},b_2a_{2j},\cdots,b_Na_{Nj})^T$ $(j=1,2,\cdots,N)$.

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